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Recent uses of connectedness in functional analysis

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Perhaps, it is not too far from the truth to say that, among the great concepts (as compactness, completeness, order, convexity) on which functional analysis is based, connectedness is relatively less popular, though this does not mean that it is less useful than the others. The aim of this lecture is just to support this latter sentence, focusing some recent results where connectedness plays a central role.

Our starting point is Theorem 1 below. Before stating it, to give the reader the convenience to realize analogies and differences, we recall, grouped together in Theorem A, three very famous results due to K.Fan and F.E.Browder.

Given a product space $X \times Y$, we denote by p_X and p_Y the projections from $X \times Y$ onto X and Y, respectively. Moreover, if $A \subseteq X \times Y$, for every $x \in X$ and $y \in Y$, we put

$$A_x = \{v \in Y : (x, v) \in A\}$$

and

$$A^y = \{u \in X : (u, y) \in A\}.$$

THEOREM A ([5], Theorems 1 and 2; [1], Theorem 7). - *Let E, F be two real Hausdorff locally convex topological vector spaces, let $X \subseteq E$, $Y \subseteq F$ be two non-empty compact convex sets, and let S, T be two subsets of $X \times Y$. Assume that at least one of the following three sets of conditions is satisfied:*
(α) *S^y is convex for each $y \in Y$, S_x is open in Y for each $x \in X$, T_x convex for each $x \in X$, and T^y is open in X for each $y \in Y$;*
(β) *S, T are closed, S^y is convex for each $y \in Y$, and T_x is convex for each $x \in X$;*
(γ) *S^y is convex for each $y \in Y$, S_x is open in Y for each $x \in X$, T is closed, and T_x is convex for each $x \in X$.*

Then, at least one of the following assertions does hold:
(a) $p_X(T) \neq X$.
(b) $p_Y(S) \neq Y$.
(c) $S \cap T \neq \emptyset$.

In [18], we pointed out that, when Y is a segment, Theorem A is still true assuming simply that the sections S^y are connected. More precisely, we have the following

THEOREM 1 ([18], Theorem 2.3). - *Let X, Y be two topological spaces, with Y admitting a continuous bijection onto $[0, 1]$, and let S, T be two subsets of $X \times Y$, with S connected and, for each $x \in X$, T_x connected. Moreover, assume that either T^y is open for each $y \in Y$, or Y is compact and T is closed.*

Then, at least one of the following assertions does hold:
(a) $p_X(T) \neq X$.

(b) $p_Y(S) \neq Y$ *and* $\{y \in Y : (p_X(S) \times \{y\}) \cap T = \emptyset\} \neq \emptyset$.
(c) $S \cap T \neq \emptyset$.

The following proposition is useful to recognize the connectedness of a given set in a product space.

PROPOSITION 1 ([18], Theorem 2.4). - *Let X, Y be two topological spaces and let S be a subset of $X \times Y$. Assume that at least one of the following four sets of conditions is satisfied:*
(γ_1) *$p_Y(S)$ is connected, S^y is connected for each $y \in Y$, and S_x is open for each $x \in X$;*
(γ_2) *$p_Y(S)$ is connected, X is compact, S is closed, and S^y is connected for each $y \in Y$;*
(γ_3) *$p_X(S)$ is connected, S_x is connected for each $x \in X$, and S^y is open for each $y \in Y$;*
(γ_4) *$p_X(S)$ is connected, Y is compact, S is closed and S_x is connected for each $x \in X$.*
Under such hypotheses, S is connected.

Then, thanks to Proposition 1, we have the following particular case of Theorem 1 which is directly comparable with Theorem A (see also [2]):

THEOREM 2 ([18], Theorem 2.5). - *Let X, Y be two topological spaces, with Y admitting a continuous bijection onto $[0, 1]$, and let S, T be two subsets of $X \times Y$. Assume that at least one of the following eight sets of conditions is satisfied:*
(δ_1) *$p_Y(S)$ is connected, S^y is connected for each $y \in Y$, S_x is open for each $x \in X$, T_x is connected for each $x \in X$, and T^y is open for each $y \in Y$;*
(δ_2) *$p_Y(S)$ is connected, Y is compact, S^y is connected for each $y \in Y$, S_x is open for each $x \in X$, T is closed, and T_x is connected for each $x \in X$;*
(δ_3) *$p_Y(S)$ is connected, X is compact, S is closed, S^y is connected for each $y \in Y$, T_x is connected for each $x \in X$, and T^y is open for each $y \in Y$;*
(δ_4) *$p_Y(S)$ is connected, X and Y are compact, S and T are closed, S^y is connected for each $y \in Y$, and T_x is connected for each $x \in X$;*
(δ_5) *$p_X(S)$ is connected, S_x and T_x are connected for each $x \in X$, and S^y and T^y are open for each $y \in Y$;*
(δ_6) *$p_X(S)$ is connected, Y is compact, S_x is connected for each $x \in X$, S^y is open for each $y \in Y$, T is closed, and T_x is connected for each $x \in X$;*
(δ_7) *$p_X(S)$ is connected, Y is compact, S is closed, S_x and T_x are connected for each $x \in X$, and T^y is open for each $y \in Y$;*
(δ_8) *$p_X(S)$ is connected, Y is compact, S and T are closed, and S_x and T_x are connected for each $x \in X$.*
Then, at least one of the following assertions does hold:
(a) $p_X(T) \neq X$.
(b) $p_Y(S) \neq Y$ *and* $\{y \in Y : (p_X(S) \times \{y\}) \cap T = \emptyset\} \neq \emptyset$.
(c) $S \cap T \neq \emptyset$.

We now start to present a series of applications of Theorems 1 and 2. The first of them concerns the following mini-max theorem:

THEOREM 3 ([18], Theorem 1.1). - *Let X, Y be two topological spaces, with Y connected and admitting a continuous bijection onto $[0, 1]$, and let f be a real function on*

$X \times Y$. *Assume that, for each* $\lambda > \sup_{y\in Y} \inf_{x\in X} f(x,y)$, $x_0 \in X$, $y_0 \in Y$, *the sets*

$$\{x \in X : f(x, y_0) \leq \lambda\}$$

and

$$\{y \in Y : f(x_0, y) > \lambda\}$$

are connected. In addition, assume that at least one of the following three sets of conditions is satisfied:
(h_1) $f(x,\cdot)$ *is upper semicontinuous in* Y *for each* $x \in X$, *and* $f(\cdot, y)$ *is lower semicontinuous in* X *for each* $y \in Y$;
(h_2) Y *is compact, and* f *is upper semicontinuous in* $X \times Y$;
(h_3) X *is compact, and* f *is lower semicontinuous in* $X \times Y$.
Under such hypotheses, one has

$$\sup_{y\in Y} \inf_{x\in X} f(x,y) = \inf_{x\in X} \sup_{y\in Y} f(x,y).$$

Two applications of Theorem 3 will be, in turn, presented later.

Another application of Theorem 2 yields the following result on the existence of Nash equilibrium points which is directly comparable with Theorem 4 of [5].

THEOREM 4 ([21], Theorem 10). - *Let* X *be a Hausdorff compact topological space,* Y *an arc, and* f, g *two continuous real functions on* $X \times Y$ *such that, for each* $\lambda \in R$, $x_0 \in X$, $y_0 \in Y$, *the sets* $\{x \in X : f(x, y_0) \geq \lambda\}$ *and* $\{y \in Y : g(x_0, y) \geq \lambda\}$ *are connected.*
Then, there exists $(x^*, y^*) \in X \times Y$ *such that*

$$f(x^*, y^*) = \max_{x\in X} f(x, y^*)$$

and

$$g(x^*, y^*) = \max_{y\in Y} g(x^*, y).$$

Another consequence of Theorem 2 is the following

THEOREM 5 ([21], Theorem 5). - *Let* E *be an infinite-dimensional Hausdorff topological vector space* E, $X \subseteq E$ *a convex set with non-empty interior,* $K \subseteq E$ *a relatively compact subset,* $Y \subseteq R$ *a compact interval, and* S, T *two subsets of* $X \times Y$. *Assume that:*
(i) S_x *is open in* Y *for each* $x \in X \setminus K$, *and* S^y *is convex and with non-empty interior for each* $y \in Y$;
(ii) T_x *is non-empty and connected for each* $x \in X \setminus K$, *and either* $T^y \setminus K$ *is open in* $X \setminus K$ *for each* $y \in Y$, *or* Y *is compact and* $T \setminus (K \times Y)$ *is closed in* $(X \setminus K) \times Y$.
Then, for every set $V \subseteq X \times Y$ *such that* V^y *is relatively compact in* E *for each* $y \in Y$ *and* V_x *is closed in* Y *for each* $x \in X \setminus K$, *the set* $(S \setminus (V \cup (K \times Y))) \cap T$ *is non-empty.*

Theorem 5 was applied in [3] by A.Chinnì to obtain what seems to be the first minimax theorem, involving two functions f, g, where it is not assumed that $f \leq g$. Her result is as follows:

THEOREM 6 ([3], Theorem 1). - *Let E, X, K, Y be as in Theorem 5, and let f, g, h be three real functions on $X \times Y$. Assume that:*
(a) $f(x, \cdot)$ *is quasi-concave in* Y *for each* $x \in X \setminus K$, *and either* f *is upper semicontinuous in* $(X \setminus K) \times Y$ *or* $f(\cdot, y)$ *is lower semicontinuous in* $X \setminus K$ *for each* $y \in Y$;
(b) $g(x, \cdot)$ *is upper semicontinuous in* Y *for each* $x \in X \setminus K$, *and* $g(\cdot, y)$ *is upper semicontinuous and quasi-convex in* X *for each* $y \in Y$;
(c) $h(x, \cdot)$ *is upper semicontinuous in* Y *for each* $x \in X \setminus K$, *and the set* $\{x \in X : h(x, y) \geq \lambda\}$ *is relatively compact in* E *for each* $y \in Y$ *and each* $\lambda > \sup_{v \in Y} \inf_{u \in X} g(u, v)$;
(d) $f(x, y) \leq \max\{g(x, y), h(x, y)\}$ *for each* $(x, y) \in (X \setminus K) \times Y$.
Then, for every relatively compact set $H \subseteq E$, *one has*

$$\inf_{x \in X \setminus H} \sup_{y \in Y} f(x, y) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y).$$

A joint application of Theorem 2 with the classical Mazurkiewicz theorem on the covering dimension, yields Theorem 7 below which could be of interest in control theory.

Precisely, let b be a positive real number and let F be a given multifunction from $[0, b] \times R^n$ into R^n. We denote by S_F the set of all Carathéodory solutions of the problem $x' \in F(t, x), x(0) = 0$ in $[0, b]$. That is to say

$$S_F = \{u \in AC([0, b], R^n) : u'(t) \in F(t, u(t)) \quad \text{a.e. in } [0, b],\ u(0) = 0\}$$

where, of course, $AC([0, b], R^n)$ denotes the space of all absolutely continuous functions from $[0, b]$ into R^n. For each $t \in [0, b]$, put

$$A_F(t) = \{u(t) : u \in S_F\}.$$

In other words, $A_F(t)$ denotes the attainable set at time t. Also, put

$$V_F = \bigcup_{t \in [0, b]} A_F(t).$$

Finally, set

$$C_F = \{x \in R^n : \{t \in [0, b] : x \in A_F(t)\} \text{ is connected}\}.$$

THEOREM 7 ([21], Theorem 9). - *Assume that* F *has non-empty compact convex values and bounded range. Moreover, assume that* $F(\cdot, x)$ *is measurable for each* $x \in R^n$ *and that* $F(t, \cdot)$ *is upper semicontinuous for a.e.* $t \in [0, b]$.
Then, for every non-empty connected set $X \subseteq V_F \cap C_F$ *which is open in its affine hull and different from* $\{0\}$, *one has the following alternative:*
either

$$X \subseteq A_F(b)$$

or

$$\dim(A_F(t) \cap X) \geq \dim(X) - 1$$

for some $t \in]0, b[$, *where* $\dim(X)$ *denotes the covering dimension of* X.

It is also worth noticing another application of Theorem 2 which allowed P.Cubiotti and B.Di Bella to get the following result, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in R^n.

THEOREM 8 ([4], Theorem 4). - *Let* $f : [0, 1] \to R^n$ $(n \geq 2)$ *be a continuous function, and let* $Y = \{y \in R^n : \| y \| = 1\}$. *Assume that, for each* $\sigma < 0$, *there exists* $L_\sigma > 0$ *such that, for each finite set* $\{y_1, ..., y_k\} \subseteq Y$, *there is a set* $\{t_1, ..., t_k\} \subseteq [0, 1]$ *such that*

$$\langle f(t_i), y_i \rangle \geq \sigma \quad \text{and} \quad |t_i - t_j| \leq L_\sigma \| y_i - y_j \|$$

for all $i, j = 1, ..., k$.

Then, f *vanishes at some point of* $[0, 1]$.

The next result comes out from a joint application of Theorem 1 with the classical Leray-Schauder continuation principle.

THEOREM 9 ([21], Theorem 12). - *Let* E *be a Banach space,* $[a, b]$ *a compact real interval,* Ω *a non-empty open bounded subset of* E, f *a continuous function from* $\overline{\Omega} \times [a, b]$ *into* E, *with relatively compact range. Assume that* $f(x, y) \neq x$ *for all* $(x, y) \in \partial\Omega \times [a, b]$ *and that the Leray-Schauder index of* $f(\cdot, a)$ *is not zero.*

Then, for every lower semicontinuous function $\varphi : \Omega \to [a, b]$ *and every upper semicontinuous function* $\psi : \Omega \to [a, b]$, *with* $\varphi(x) \leq \psi(x)$ *for all* $x \in \Omega$, *there exist* $x^* \in \Omega$ *and* $y^* \in [\varphi(x^*), \psi(x^*)]$ *such that* $f(x^*, y^*) = x^*$.

In addition, if for some sequence $\{\lambda_n\}$ *of positive real numbers, with* $\inf_{n \in N} \lambda_n = 0$, *one has*

$$\inf\{y \in [a, b] : \| f(x, y) - x \| \geq \lambda_n\} = \inf\{y \in [a, b] : \| f(x, y) - x \| > \lambda_n\}$$

for each $x \in \Omega$, $n \in N$ *for which*

$$\{y \in [a, b] : \| f(x, y) - x \| > \lambda_n\} \neq \emptyset,$$

then there exists $x_0 \in \Omega$ *such that* $f(x_0, y) = x_0$ *for all* $y \in [a, b]$.

We now come to the two announced applications of Theorem 3. The first of them is due to O.Naselli ([8]). Making use of Theorem 3, she got, as a corollary of a more general result, the following

THEOREM 10 ([8], Theorem 3.4). - *Let* E *be a real Hausdorff topological vector space,* p *a real number greater than* 1, *and* α, β, γ *three affine functionals on* E, *with* $\gamma(0) \geq 0$.

Then, for every closed, bounded and convex set $X \subseteq \gamma^{-1}([\gamma(0), +\infty[) \cap \gamma^{-1}(]0, +\infty[)$, *with* $\dim(X) \geq 2$, *one has*

$$\inf_{x \in X} \left(\alpha(x) + \left(\frac{|\beta(x)|^p}{\gamma(x)} \right)^{\frac{1}{p-1}} \right) = \inf_{x \in B_X} \left(\alpha(x) + \left(\frac{|\beta(x)|^p}{\gamma(x)} \right)^{\frac{1}{p-1}} \right),$$

where

$$B_X = \{x \in X : \exists y \in \mathrm{aff}(X) \setminus \{x\} : [x, y] \cap X = \{x\}\},$$

aff(X) *being the affine hull of* X, *and* $[x, y]$ *being the line segment joining* x *and* y.

The other application of Theorem 3 we wish to recall concerns integral functionals. We first introduce some notation.

In the next four results, (T, F, μ) is a σ-finite non-atomic measure space ($\mu(T) > 0$), $(E, \|\cdot\|)$ is a real Banach space ($E \neq \{0\}$), and p is a real number greater than or equal to 1. When $p = 1$, we will adopt the convention $\frac{p}{p-1} = \infty$.

For simplicity, we denote by X the usual space $L^p(T, E)$ of (equivalence classes of) strongly μ-measurable functions $u : T \to E$ such that $\int_T \| u(t) \|^p d\mu < +\infty$, equipped with the norm $\| u \|_X = (\int_T \| u(t) \|^p d\mu)^{\frac{1}{p}}$.

Moreover, we denote by $V(X)$ the family of all sets $V \subseteq X$ of the following type:

$$V = \{u \in X : \Psi(u) = \int_T g(t, u(t))d\mu\}$$

where Ψ is a continuous linear functional on X, and $g : T \times E \to R$ is such that the integral functional $u \to \int_T g(t, u(t))d\mu$ is (well-defined and) Lipschitzian in X, with Lipschitz constant strictly less than $\| \Psi \|_{X^*}$.

Note, in particular, that each closed hyperplane of X belongs to the family $V(X)$.

We then have

THEOREM 11 ([22], Theorem 2). - *Let* $f : T \times E \to [0, +\infty[$ *be such that* $f(\cdot, x)$ *is* μ*-measurable for each* $x \in E$ *and* $f(t, \cdot)$ *is Lipschitzian with Lipschitz constant* $M(t)$ *for almost every* $t \in T$, *where* $M \in L^{\frac{p}{p-1}}(T)$. *Assume that* $f(\cdot, 0) \in L^1(T)$ *and that there exists a sequence* $\{\lambda_n\}$ *in* $]0, +\infty[$, *with* $\lim_{n\to+\infty} \lambda_n = +\infty$, *such that, for almost every* $t \in T$ *and for every* $x \in E$, *one has*

$$\lim_{n\to+\infty} \frac{f(t, \lambda_n x)}{\lambda_n} = 0.$$

Then, for every $V \in V(X)$, *one has*

$$\inf_{u\in V} \int_T f(t, u(t))d\mu = \inf_{u\in X} \int_T f(t, u(t))d\mu.$$

The proof of Theorem 11 is fully based on an application of Lemma 1 of [19]. It is just this latter to be obtained by means of an application of Theorem 3. It is also worth noticing that such an application is made possible by the following very interesting result of J.Saint Raymond:

THEOREM 12 ([23], Théorème 3). - *Let* $f : T \times E \to R$ *be a* $F \otimes B(E)$*-measurable function,* $B(E)$ *being the Borel family of* E. *Then, if we put*

$$Y = \{u \in X : f(\cdot, u(\cdot)) \in L^1(T)\},$$

for each $\lambda \in R$. *the set*

$$\{u \in Y : \int_T f(t, u(t))d\mu \leq \lambda\}$$

is connected.

Theorem 11 has the following two consequences.

THEOREM 13 ([22], Theorem 1). - *Let* E *be separable, and let* $F : T \to 2^E$ *be a measurable multifunction, with non-empty closed values. Assume that* $\operatorname{dist}(0, F(\cdot)) \in L^1(T)$ *and that there exists a sequence* $\{\lambda_n\}$ *in* $]0, +\infty[$. *with* $\lim_{n\to+\infty} \lambda_n = +\infty$. *such that, for almost every* $t \in T$ *and for every* $x \in E$, *one has*

$$\lim_{n\to+\infty} \frac{\operatorname{dist}(\lambda_n x, F(t))}{\lambda_n} = 0.$$

Then, if $p = 1$, *each member of the family* $V(X)$ *contains a selection of* F.

THEOREM 14 ([22], Theorem 6). - *Let* E *be reflexive and separable, let* $p > 1$, *and let* $f : T \times E \to [0, +\infty[$ *be such that* $f(\cdot, x)$ *is* μ*-measurable for each* $x \in E$, $f(\cdot, 0) \in L^1(T)$, *and* $f(t, \cdot)$ *is Gâteaux differentiable for almost every* $t \in T$. *Moreover, assume that there exist* $M \in L^{\frac{p}{p-1}}(T)$ *and a sequence* $\{\lambda_n\}$ *in* $]0, +\infty[$, *with* $\lim_{n\to+\infty} \lambda_n = +\infty$, *such that, for almost every* $t \in T$ *and for every* $x \in E$, *one has*

$$\| f'_x(t, x) \|_{E^*} \leq M(t)$$

and

$$\lim_{n\to+\infty} \frac{f(t, \lambda_n x)}{\lambda_n} = 0.$$

Then, for every $V \in V(X)$, *there exists a sequence* $\{u_n\}$ *in* V *such that*

$$\lim_{n\to+\infty} \int_T f(t, u_n(t))d\mu = \inf_{u\in X} \int_T f(t, u(t))d\mu$$

and

$$\lim_{n\to+\infty} \int_T \| f'_x(t, u_n(t)) \|_{E^*}^{\frac{p}{p-1}} d\mu = 0.$$

The final part of our lecture is devoted to recent applications of the following lower semicontinuity result, based itself on connectedness:

THEOREM 15 ([10], Théorème 1.1). - *Let* X, Y *be two topological spaces, with* Y *connected and locally connected, and let* $\varphi : X \times Y \to R$ *be a function satisfying the following two conditions:*

(a) *for each* $x \in X$, *the function* $\varphi(x, \cdot)$ *is continuous,* $0 \in \operatorname{int}(\varphi(x, Y))$, *and* $\operatorname{int}(\{y \in Y : \varphi(x, y) = 0\}) = \emptyset$;

(b) *the set*

$$\{(y, z) \in Y \times Y : \{x \in X : \varphi(x, y) < 0 < \varphi(x, z)\} \text{ is open}\}$$

is dense in $Y \times Y$.

Then, if, for each $x \in X$, *one denotes by* $Q(x)$ *the set of all* $y \in Y$ *such that* $\varphi(x, y) = 0$ *and* y *is not a local extremum for* $\varphi(x, \cdot)$, *one has that* $Q(x)$ *is non-emtpy and closed, and that the multifunction* $x \to Q(x)$ *is lower semicontinuous.*

We now recall two applications of Theorem 15 to implicit differential equations.

THEOREM 16 ([17], Theorem 2). - *Let* Y *be a linear subspace of* R^n, *with* $\dim(Y) \geq 2$, *and let* $f : [0, 1] \times R^{nk} \times Y \to R$ *be a continuous functions such that, for each* $(t, \xi) \in [0, 1] \times R^{nk}$, $f(t, \xi, \cdot)$ *is affine and non-constant in* Y.

Then, for every $x_0, x_1, ..., x_{k-1} \in R^n$, *there exists* $b \in]0, 1]$ *such that the set of all functions* $u \in C^k([0, b], R^n)$ *satisfying*

$$u^{(k)}(t) \in Y, \ f(t, u(t), u'(t), ..., u^{(k)}(t)) = 0 \ \ in \ \ [0, b],$$

$$u^{(i)}(0) = x_i \ \ for \ \ i = 0, 1, ..., k - 1,$$

has the continuum power.

THEOREM 17 ([7], Example 4.1). - *Let* $\Omega \subseteq R^n$ ($n \geq 3$) *be an open, bounded, connected subset, with a boundary of class* $C^{1,1}$.

Then, for every $g \in L^p(\Omega)$, *with* $p \in]n, +\infty[$, $\gamma \in [0, 1[$, $\lambda, \mu \in R$, *there exists* $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *such that*

$$\Delta u(x) = \lambda \sin \Delta u(x) + \mu(|u(x)| + \| \nabla u(x) \|)^\gamma + g(x)$$

for almost every $x \in \Omega$.

For other papers related to Theorem 15, we refer to [9], [11], [12], [13], [14], [15].

Before establishing the final applications of Theorem 15, we also recall the following

THEOREM 18 ([16], Théorème 2). - *Let* X, Y *be two real Banach spaces, let* $\Phi : X \to Y$ *be a continuous linear surjective operator, and let* $\Psi : X \to Y$ *be a Lipschitzian operator, with Lipschitz constant* $L < \frac{1}{\alpha_\Phi}$, *where* $\alpha_\Phi = \sup_{\|y\| \leq 1} \mathrm{dist}(0, \Phi^{-1}(y))$.

Then, for each $y \in Y$, *the set* $(\Phi + \Psi)^{-1}(y)$ *is a (non-empty) retract of* X, *and the multifunction* $y \to (\Phi + \Psi)^{-1}(y)$ *is Lipschitzian (with respect to the Hausdorff distance), with Lipschitz constant* $\frac{\alpha_\Phi}{1 - L\alpha_\Phi}$.

We now can prove

THEOREM 19. - *Let* X *be a connected topological space,* E *a real Banach space (with topological dual space* E^**),* Φ *an operator from* X *into* E^*, f *a real function on* $X \times E$ *such that, for each* $x \in X$, $f(x, \cdot)$ *is Lipschitzian in* E, *with Lipschitz constant* $L(x) \geq 0$. *Further, assume that the set*

$$\{y \in E : \langle \Phi(\cdot), y \rangle - f(\cdot, y) \ \ is \ \ continuous\}$$

is dense in E *and that the set*

$$\{(x, y) \in X \times E : \langle \Phi(x), y \rangle = f(x, y)\}$$

is disconnected.

Then, there exists some $x_0 \in X$ *such that* $\| \Phi(x_0) \|_{E^*} \leq L(x_0)$.

PROOF. Arguing by contradiction, assume that $\| \Phi(x) \|_{E^*} > L(x)$ for all $x \in X$. Then, by Theorem 18, for each $x \in X$, the function $\langle \Phi(x), \cdot \rangle - f(x, \cdot)$ is onto R, is open and has connected point inverses. At this point, we can apply Theorem 15, to get that the multifunction $Q : X \to 2^E$ defined by

$$Q(x) = \{y \in E : \langle \Phi(x), y \rangle = f(x, y)\}$$

is lower semicontinuous. Then, since X is connected and each $Q(x)$ is non-empty and connected, Theorem 3.2 of [6] ensures that the graph of Q is connected too, against one of our assumptions. $\triangle$

Observe that when, in Theorem 19, f does not depend on y (that is, $L(x) = 0$ for all $x \in X$) we directly get the existence of a zero for the operator Φ. In this case, one can even assume that E is simply a topological vector space (see [20]). To get a zero for Φ allowing f to depend on y, we can use

THEOREM 20. - *Let* X *be a connected topological space,* E *a real Banach space (with topological dual space* E^**),* Φ *an operator from* X *into* E^*, *with closed range. Assume that, for each* $\epsilon > 0$, *there exists a real function* f_ϵ *on* $X \times E$ *having the following properties:*
(a) *for each* $x \in X$, *the function* $f_\epsilon(x, \cdot)$ *is Lipschitzian in* E, *with Lipschitz constant less than or equal to* ϵ*;*
(b) *the set*

$$\{y \in E : \langle \Phi(\cdot), y \rangle - f_\epsilon(\cdot, y) \text{ is continuous}\}$$

is dense in E*;*
(c) *the set*

$$\{(x, y) \in X \times E : \langle \Phi(x), y \rangle = f_\epsilon(x, y)\}$$

is disconnected.

Then, Φ *vanishes at some point of* X.

PROOF. Applying Theorem 19, for each $\epsilon > 0$, we get a point $x_\epsilon \in X$ such that $\| \Phi(x_\epsilon) \|_{X^*} \leq \epsilon$. In other words, 0 is in the closure of $\Phi(X)$. But, by assumption, $\Phi(X)$ is closed, and so $0 \in \Phi(X)$, as claimed. $\triangle$

THEOREM 21. - *Let* X *be a connected and locally connected topological space,* E *a real Banach space,* $\Phi : X \to E^*$ *a (strongly) continuous operator,* L *a non-negative real function on* X. *Denote by* Λ *the set of all continuous functions* $f : X \times E \to R$ *such that, for each* $x \in X$, $f(x, \cdot)$ *is Lipschitzian in* E, *with Lipschitz constant less than or equal to* $L(x)$. *Consider* Λ *equipped with the relativization of the strongest vector topology on the space* $R^{X \times E}$, *and assume that the set*

$$\{(f, x, y) \in \Lambda \times X \times E : \langle \Phi(x), y \rangle = f(x, y)\}$$

is disconnected.

Then, there exists some $x_0 \in X$ *such that* $\| \Phi(x_0) \|_{E^*} \leq L(x_0)$.

PROOF. Arguing by contradiction, assume that $\| \Phi(x) \|_{E^*} > L(x)$ for all $x \in X$. For each $(f, x, y) \in \Lambda \times X \times E$, put

$$\varphi(f, x, y) = \langle \Phi(x), y \rangle - f(x, y).$$

Observe that, for each $(x, y) \in X \times E$, the function $\varphi(\cdot, x, y)$ is continuous in Λ since it is continuous even with respect to the topology of pointwise convergence. Moreover, for each $f \in \Lambda$, the function $\varphi(f, \cdot, \cdot)$ is continuous in $X \times E$ (thanks to the strong continuity of Φ), and, again by Theorem 18, is onto R, and has no local extrema. Consequently, again by Theorem 15, the multifunction $Q : \Lambda \to 2^{X \times E}$ defined by

$$Q(f) = \{(x, y) \in X \times E : \langle \Phi(x), y \rangle = f(x, y)\}$$

is lower semicontinuous. But, by Theorem 19, each set $Q(f)$ is connected. On the other hand, Λ is connected (since it is convex), and so, by Theorem 3.2 of [6], the graph of Q is connected, against one of our assumptions. $\triangle$

From Theorem 21, we get, of course, the following

THEOREM 22. - *Let X be a connected and locally connected topological space, E a real Banach space, $\Phi : X \to E^*$ a (strongly) continuous operator, with closed range. For each $\epsilon > 0$, denote by Λ_ϵ the set of all continuous functions $f : X \times E \to R$ such that, for each $x \in X$, $f(x, \cdot)$ is Lipschitzian in E, with Lipschitz constant less than or equal to ϵ. Consider Λ_ϵ equipped with the relativization of the strongest vector topology on the space $R^{X \times E}$, and assume that the set*

$$\{(f, x, y) \in \Lambda_\epsilon \times X \times E : \langle \Phi(x), y \rangle = f(x, y)\}$$

is disconnected.

Then, Φ vanishes at some point of X.

We conclude by proposing the following conjecture:

CONJECTURE. *Let X be the closed unit ball of R^n, $(n \geq 2)$, $g : X \to X$ a continuous function and $\epsilon > 0$. Denote by Λ_ϵ the set all continuous functions $f : X \times R^n \to R$ such that, for each $x \in X$, $f(x, \cdot)$ is Lipschitzian in R^n, with Lipschitz constant less than or equal to ϵ. Consider Λ_ϵ equipped with the relativization of the strongest vector topology on the space $R^{X \times R^n}$. Then, the set*

$$\{(f, x, y) \in \Lambda_\epsilon \times X \times R^n : \langle g(x) - x, y \rangle = f(x, y)\}$$

is disconnected.

Observe that, on the basis of Theorem 22, the above conjecture could lead to a completely new way of proving the Brouwer fixed point theorem.

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